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Iwasawa effects in multilayer optics

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There are many two-by-two matrices in layer optics. It is shown that they can be formulated in terms of a three-parameter group whose algebraic property is the same as the group of Lorentz transformations in a space with two spacelike and one timelike dimensions, or the $Sp(2)$ group which is a standard theoretical tool in optics. Among the interesting mathematical properties of this group, the Iwasawa decomposition drastically simplifies the matrix algebra under certain conditions, and leads to a concise expression for the S matrix for transmitted and reflected waves. It is shown that the Iwasawa effect can be observed in multilayer optics, and a sample calculation of the S matrix is given.

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I. INTRODUCTION

In a series of recent papers [1,2], Han, Kim, and Noz have formulated polarization optics in terms of the two-by-two and four-by-four representations of the six-parameter Lorentz group. They noted that the Lorentz group properties can be found in optical materials. Indeed, there are many two-by-two matrices in layer optics [3–5]. In this paper, we reorganize them within the framework of the Lorentz group. We then derive a mathematical relation that can be tested experimentally. If a light wave hits a flat surface, a part of this wave becomes reflected and the remaining part becomes transmitted.

If there are multilayers, this process repeats itself at each boundary. There has been a systematic approach to this problem based on the two-by-two S-matrix formalism [3–5]. This S matrix consists of boundary and phase-shift matrices. The phase-shift matrices are complex and the S matrix is in general complex.

However, in this paper we first show these complex matrices can be systematically transformed into a set of real unimodular (with determinant $=1$) matrices with three independent parameters. Then we can use the well-established mathematical procedure for them. This procedure is called the $Sp(2)$ group whose algebraic property is the same as that of the $SU(1,1)$ group which occupies a prominent place in optics from squeezed states of light [6]. However, the most pleasant aspect of the $Sp(2)$ group is that its algebras consist only of two-by-two matrices with real elements. When applied to a two-dimensional plane, they produce rotations and squeeze transformations [7].

It is known that these simple matrices produce some nontrivial mathematical results, namely Wigner rotations and Iwasawa decompositions [8]. The Wigner rotation means a rotation resulting from a multiplication of three squeeze matrices, and the Iwasawa decomposition means that a product of squeeze and rotation matrices, under certain conditions, leads to a matrix with one vanishing off-diagonal element. This leads to a substantial simplification in mathematics and eventually leads to a more transparent comparison of theory with experiments. This decomposition has been discussed in the literature in connection with polarization optics [9,10]. In this paper we study applications of this mathematical device in layer optics.

There are papers in the literature on applications of the Lorentz group in layer optics [2,11], but these papers are concerned with polarization optics. In this paper we are dealing with reflections and transmissions of optical waves. We show that layers with alternate indexes of refraction can exhibit an Iwasawa effect and provide a calculation of the transmission and reflection coefficients. It is remarkable that the Lorentz group can play as the fundamental scientific language even in the physics of reflections and transmissions.

In Sec. II we formulate the problem in terms of the S-matrix method widely used in optics [3]. In Sec. III this S-matrix formalism is translated into the mathematical framework of the $Sp(2)$ group consisting of two-by-two unimodular matrices with real elements. We demonstrate that there is a subset of these matrices with one vanishing nondiagonal element. It is shown possible to produce this set of matrices from multiplications of the matrices in the original set. This is called the Iwasawa decomposition. In Sec. IV we transform the mathematical formalism of the Iwasawa decomposition into the real world, and calculate the reflection and transmission coefficients which can be measured in optics laboratories.

Even though the present paper is based on some group-theoretical theorems, we used the algebra of two-by-two matrices throughout the paper while avoiding the formal mathematical language. In the Appendix we explain what we do in terms of group theory.

II. FORMULATION OF THE PROBLEM

Let us start with the S-matrix formalism of the layer optics. We start with a plane wave traveling in a given direction. If the wave is incident on a plane boundary of a me-

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dium with a different index of refraction, the problem can be formulated in terms of two-by-two matrices [3,5]. If we write the column vectors

$$\begin{pmatrix} E_1^{(+)} \\ E_1^{(-)} \end{pmatrix}, \quad \begin{pmatrix} E_2^{(+)} \\ E_2^{(-)} \end{pmatrix} \tag{1}$$

for the incident, with superscript $(+)$, and reflected, with superscript $(-)$, for the waves in the first and second media, respectively, then they are connected by the two-by-two S matrix,

$$\begin{pmatrix} E_1^{(+)} \\ E_1^{(-)} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} E_2^{(+)} \\ E_2^{(-)} \end{pmatrix}. \tag{2}$$

Of course the elements of the above S matrix depend on reflection and transmission coefficients [3].

Let us consider a light-wave incident on a flat surface, then it is decomposed into transmitted and reflected waves. If $E_1^{(+)}$ is the incident wave, the transmitted wave is $E_2^{(+)}$, with

$$E_2^{(+)} = t_{12} E_1^{(+)}, \quad E_1^{(-)} = r_{12} E_1^{(+)}. \tag{3}$$

Thus, the S matrix takes the form [3]

$$\begin{pmatrix} E_1^{(+)} \\ E_1^{(-)} \end{pmatrix} = \begin{pmatrix} 1/t_{12} & r_{12}/t_{12} \\ r_{12}/t_{12} & 1/t_{12} \end{pmatrix} \begin{pmatrix} E_2^{(+)} \\ 0 \end{pmatrix}. \tag{4}$$

If the wave comes from the second medium in the opposite direction, the same matrix can be used for

$$\begin{pmatrix} 0 \\ E_1^{(-)} \end{pmatrix} = \begin{pmatrix} 1/t_{12} & r_{12}/t_{12} \\ r_{12}/t_{12} & 1/t_{12} \end{pmatrix} \begin{pmatrix} E_2^{(+)} \\ E_2^{(-)} \end{pmatrix}. \tag{5}$$

Since the magnitude of the reflection coefficient is smaller than one, and since $t_{12}^2 + r_{12}^2 = 1$, we can write the above matrix as

$$\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}, \tag{6}$$

with

$$r_{12} = \tanh\eta, \quad t_{12} = 1/\cosh\eta. \tag{7}$$

Since this describes both the reflection and transmission at the boundary, we shall call this matrix the "boundary matrix" [12]. The reflection and transmission coefficients are, of course, derivable from Maxwell's equations with boundary conditions. The mathematics of this form is well known. It can perform Lorentz boosts when applied to the longitudinal and timelike coordinates. Recently, it has been observed that it performs squeeze transformations when applied to the two-dimensional space of x and y [7].

Next, if the wave travels within a given medium from one inner-surface to the other surface [3],

$$\begin{pmatrix} E_a^{(+)} \\ E_a^{(-)} \end{pmatrix} = \begin{pmatrix} e^{-i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} E_b^{(+)} \\ E_b^{(-)} \end{pmatrix}, \tag{8}$$

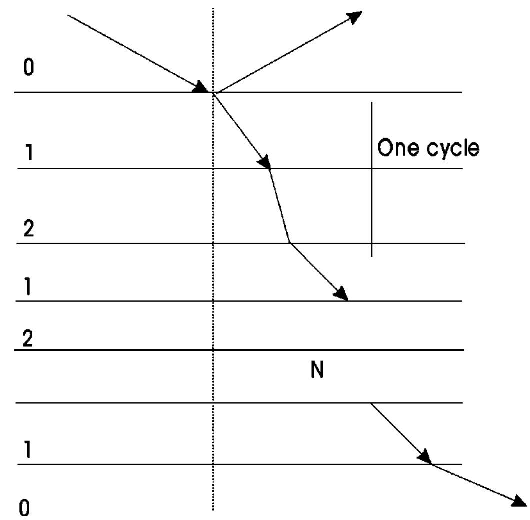

FIG. 1. Multilayer system. A light wave is incident on the first boundary, with transmitted and reflected waves. The transmitted wave goes through the first medium and hits the second medium again with reflected and transmitted waves. The transmitted wave goes through the second medium and hits the first medium. This cycle continues N times.

where the subscripts a and b are for the initial and final surfaces, respectively. The above expression tells there is a phase difference of 2δ between the waves. This phase difference depends on the index of refraction, wavelength, and the angle of incidence [3].

In this paper we consider a multilayer system, consisting of two media with different indexes of refraction as is illustrated in Fig. 1. Then, the system consists of many boundaries and phase-shift matrices. After multiplication of all those matrices, the result will be one two-by-two matrix that we introduced as the S matrix in Eq. (2). We are interested in this paper when this matrix takes special forms that can be readily tested experimentally.

If the wave hits the first medium from the air, as is illustrated in Fig. 1, we write the matrix as

$$\begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}. \tag{9}$$

Within the first medium, the phase-shift matrix becomes

$$\begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}. \tag{10}$$

When the wave hits the surface of the second medium, the corresponding matrix is

$$\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}. \tag{11}$$

Within the second medium, we write the phase-shift matrix as

$$\begin{pmatrix} e^{-i\xi} & 0 \\ 0 & e^{i\xi} \end{pmatrix}. \tag{12}$$

Then, when the wave hits the first medium from the second

$$\begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix}. \tag{13}$$

But if the thickness of the first medium is zero, and the wave exists to the air, then the system goes through the boundary matrix

$$\begin{pmatrix} \cosh\lambda & -\sinh\lambda \\ -\sinh\lambda & \cosh\lambda \end{pmatrix}. \tag{14}$$

The net result is

$$\begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \cosh\lambda & -\sinh\lambda \\ -\sinh\lambda & \cosh\lambda \end{pmatrix}, \tag{15}$$

with

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}\begin{pmatrix} e^{-i\xi} & 0 \\ 0 & e^{i\xi} \end{pmatrix} \times \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix}. \tag{16}$$

If the wave goes through N cycles of this pair of layers, the S matrix becomes

$$\begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^N\begin{pmatrix} \cosh\lambda & -\sinh\lambda \\ -\sinh\lambda & \cosh\lambda \end{pmatrix}. \tag{17}$$

Thus, the problem reduces to looking into unusual properties of the core matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^N. \tag{18}$$

We realize that the numerical computation of this expression is rather trivial these days, but we are still interested in the mathematical form which takes an exceptionally simple form. It is still an interesting problem to produce mathematics that enable us to perform calculations without using computers. In Sec. III we shall consider mathematical simplification coming from one vanishing off-diagonal element.

III. MATHEMATICAL INSTRUMENT

The core matrix of Eq. (18) contains the chain of the matrices

$$W = \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}\begin{pmatrix} e^{-i\xi} & 0 \\ 0 & e^{i\xi} \end{pmatrix}. \tag{19}$$

The Lorentz group allows us to simplify this expression under certain conditions.

For this purpose we transform the above expression into a more convenient form by taking the conjugate of each of the matrices with

$$C_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{20}$$

Then $C_1 W C_1^{-1}$ leads to

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}\begin{pmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{pmatrix}. \tag{21}$$

In this way, we have converted W of Eq. (19) into a real matrix, but it is not simple enough.

Let us take another conjugate with

$$C_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{22}$$

Then the conjugate $C_2 C_1 W C_1^{-1} C_2^{-1}$ becomes

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} e^{\eta} & 0 \\ 0 & e^{-\eta} \end{pmatrix}\begin{pmatrix} \cos\xi & -\sin\xi \\ \sin\xi & \cos\xi \end{pmatrix}. \tag{23}$$

The combined effect of $C_2 C_1$ is

$$C = C_2 C_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\pi/4} & e^{i\pi/4} \\ -e^{-i\pi/4} & e^{-i\pi/4} \end{pmatrix}, \tag{24}$$

with

$$C^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\pi/4} & -e^{i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{pmatrix}. \tag{25}$$

After multiplication, the matrix of Eq. (23) will take the form

$$V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{26}$$

where A, B, C, and D are real numbers. If B and C vanish, this matrix will becomes diagonal, and the problem will become too simple. If, on the other hand, only one of these two elements become zero, we will achieve a substantial mathematical simplification and will be encouraged to look for physical circumstances that will lead to this simplification.

Let us summarize. We started in this section with the matrix representation W given in Eq. (19). This form can be transformed into the V matrix of Eq. (23) through the conjugate transformation

$$V = CWC^{-1}, \tag{27}$$

where C is given in Eq. (24). Conversely, we can recover the W representation by

$$W = C^{-1}VC. \tag{28}$$

For calculational purposes, the V representation is much easier because we are dealing with real numbers. On the other hand, the W representation is of the form for the S matrix we intend to compute. It is gratifying to see that they are equivalent.

Let us go back to Eq. (23) and consider the case where the angles ϕ and ξ satisfy the following constraints:

$$\phi+\xi=2\theta, \quad \phi-\xi=\pi/2, \tag{29}$$

thus

$$\phi=\theta+\pi/4, \quad \xi=\theta-\pi/4. \tag{30}$$

Then in terms of θ, we can reduce the matrix of Eq. (23) to the form

$$\begin{pmatrix} (\cosh\eta)\cos(2\theta) & \sinh\eta-(\cosh\eta)\sin(2\theta) \\ \sinh\eta+(\cosh\eta)\sin(2\theta) & (\cosh\eta)\cos(2\theta) \end{pmatrix}. \tag{31}$$

Thus the matrix takes a surprisingly simple form if the parameters θ and η satisfy the constraint

$$\sinh\eta=(\cosh\eta)\sin(2\theta). \tag{32}$$

Then the matrix becomes

$$\begin{pmatrix} 1 & 0 \\ 2\sinh\eta & 1 \end{pmatrix}. \tag{33}$$

This aspect of the Lorentz group is known as the Iwasawa decomposition [8], and has been discussed in the optics literature [9,10].

The matrices of the form is not so strange in optics. In para-axial lens optics, the translation and lens matrices are written as

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}, \tag{34}$$

respectively. These matrices have the following interesting mathematical property [2]:

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & u_1+u_2 \\ 0 & 1 \end{pmatrix} \tag{35}$$

and

$$\begin{pmatrix} 1 & 0 \\ u_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u_1 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ u_1+u_2 & 1 \end{pmatrix}. \tag{36}$$

We note that the multiplication is commutative, and the parameter becomes additive. These matrices convert multiplication into addition, as logarithmic functions do.

Throughout this section we used the algebra of two-by-two matrices, while avoiding formal group-theoretical languages. In the Appendix, we give a group-theoretical interpretation of what we are doing in this paper.

IV. POSSIBLE EXPERIMENTS

The question then is whether it is possible to construct optical layers that will perform this or similar calculation. In order to make contacts with the real world, let us extend the algebra to the form

$$\begin{pmatrix} 1 & 0 \\ 2\sinh\eta & 1 \end{pmatrix}\begin{pmatrix} e^{-\eta} & 0 \\ 0 & e^{\eta} \end{pmatrix}, \tag{37}$$

which becomes

$$\begin{pmatrix} e^{-\eta} & 0 \\ 2e^{-\eta}\sinh\eta & e^{\eta} \end{pmatrix}. \tag{38}$$

The square of this matrix is

$$\begin{pmatrix} e^{-\eta} & 0 \\ 2e^{-\eta}\sinh\eta & e^{\eta} \end{pmatrix}^2=\begin{pmatrix} e^{-2\eta} & 0 \\ 2(e^{-2\eta}+1)\sinh\eta & e^{2\eta} \end{pmatrix}. \tag{39}$$

If we repeat this process,

$$\begin{pmatrix} e^{-\eta} & 0 \\ 2e^{-\eta}\sinh\eta & e^{\eta} \end{pmatrix}^N=\begin{pmatrix} e^{N\eta} & 0 \\ 2b(\sinh\eta) & e^{-N\eta} \end{pmatrix}, \tag{40}$$

with

$$b=e^{-N\eta}\sum_{k=1}^{N-1} e^{-2(k-1)\eta}, \tag{41}$$

which can be simplified to

$$b=\frac{e^{-\eta}\sinh(N\eta)}{\sinh\eta}. \tag{42}$$

Then we can write Eq. (40) as

$$\begin{pmatrix} e^{-\eta} & 0 \\ 2e^{-\eta}\sinh\eta & e^{\eta} \end{pmatrix}^N=\begin{pmatrix} e^{-N\eta} & 0 \\ 2e^{-\eta}\sinh(N\eta) & e^{N\eta} \end{pmatrix}. \tag{43}$$

If we take into account the boundary between the air and the first medium,

$$\begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}\begin{pmatrix} e^{-N\eta} & 0 \\ 2e^{-\eta}\sinh(N\eta) & e^{N\eta} \end{pmatrix}\begin{pmatrix} e^{-\lambda} & 0 \\ 0 & e^{\lambda} \end{pmatrix} \\ =\begin{pmatrix} e^{-N\eta} & 0 \\ 2e^{-(2\lambda+\eta)}\sinh(N\eta) & e^{N\eta} \end{pmatrix}. \tag{44}$$

Thus, the original matrix of Eq. (2) becomes

$$\begin{pmatrix} \cosh(N\eta)+ie^{-(\eta+2\lambda)}\sinh(N\eta) & -(1+ie^{-(\eta+2\lambda)})\sinh(N\eta) \\ -(1-ie^{-(\eta+2\lambda)})\sinh(N\eta) & \cosh(N\eta)-ie^{-(\eta-2\lambda)}\sinh(N\eta) \end{pmatrix}. \tag{45}$$

From the S-matrix formalism, the reflection and transmission coefficients are

$$R=\frac{E_a^{(-)}}{E_a^{(+)}}=\frac{S_{21}}{S_{11}},$$
$$T=\frac{E_s^{(+)}}{E_a^{(+)}}=\frac{1}{S_{21}}. \tag{46}$$

Thus, they become

$$R=\frac{(1-ie^{-(\eta+2\lambda)})\sinh(N\eta)}{\cosh(N\eta)+ie^{-(\eta+2\lambda)}\sinh(N\eta)},$$
$$T=\frac{-1}{(1-ie^{-(\eta+2\lambda)})\sinh(N\eta)}. \tag{47}$$

The above expression depends only the number of layer cycles N and the parameter η, which was defined in terms of the reflection and transmission coefficients in Eq. (7). It is important also that the above simple form is possible only if the phase-shift parameters ϕ and ξ should satisfy the relations given in Eqs. (30) and (32). In summary, they should satisfy

$$\cos(2\xi)=-\cos(2\phi) \quad \text{and} \quad \tanh\eta=\cos(2\xi). \tag{48}$$

In setting up the experiment, we note that all three parameters η, ξ, and ϕ depend on the incident angle and the frequency of the light wave. The parameter η is derivable from the reflection and transmission coefficients which depend on both the angle and frequency. The angular parameters ξ and ϕ depend on the optical path and the index of refraction which depend on the incident angle and the frequency, respectively.

Now all three quantities in Eq. (48) are functions of the incident angle and the frequency. If we consider a three-dimensional space with the incident angle and frequency as the x and y axes, respectively, all three quantities, $\cos(2\xi)$, $\cos(2\phi)$, and $\tanh\eta$, will be represented by two-dimensional surfaces. If we choose $\cos(2\xi)$ and $\cos(2\phi)$, the intersection will be a line. This line will pass through the third surface for $\tanh\eta$. The point at which the line passes through the surface corresponds to the values of the incident angle and frequency which will satisfy the two conditions given in Eq. (48).

While it is possible to set up this experiment, it will require computer work to determine a point where the three planes coincide at one point. It does not take too much additional work to compute the S matrix without the Iwasawa effects if the number of layers is not large. The computer can handle the problem easily if N is about 10, 100, or even 1000. It would indeed be interesting how this Iwasawa effect stands out from the computer calculation.

V. CONCLUDING REMARKS

In this paper we borrowed the concept of Iwasawa decomposition from well-known theorems in group theory. On the other hand, group theory appears in this paper in the form of two-by-two matrices with three independent parameters. The Iwasawa decomposition makes the algebra of two-by-two matrices even simpler. It is interesting to note that there still is room for mathematical simplifications in the algebra of two-by-two matrices and that this procedure can be tested in optics laboratories.

APPENDIX: FURTHER MATHEMATICAL DETAILS

In Sec. III, which contains the mathematical instrument for this paper, we restricted ourselves to the algebra of two-by-two matrices and avoided as much as possible group-theoretical languages. In order to explain where those algebraic tricks came from, we give in this appendix a group-theoretical interpretation of what we did in this paper.

The group $SL(2,c)$ consists of two-by-two unimodular matrices whose elements are complex. There are therefore six independent parameters, and thus six generators of the Lie algebra. This group is locally isomorphic to the six-parameter Lorentz group or $O(3,1)$ applicable to the Minkowskian space of three spacelike directions and one timelike direction.

Like the Lorentz group, the $SL(2,c)$ has a number of interesting subgroups. The subgroup most familiar to us is $SU(2)$ which is locally isomorphic to the three-dimensional rotation group. In addition, this group contains three subgroups that are locally isomorphic to the group $O(2,1)$ applicable to the Minkowskian space of two spacelike and one timelike dimensions.

One of the subgroups of $SL(2,c)$ is $SL(2,r)$ consisting of matrices with real elements. This subgroup is also called the $Sp(2)$ group which we used in this paper in order to carry out the Iwasawa decomposition. Another interesting subgroup is the one we used for computing the S matrix, which starts with the boundary matrix of Eq. (6) and the phase-shift matrix of Eq. (8). This group is called $SU(1,1)$. The present paper exploits the isomorphism between $Sp(2)$ and $SU(1,1)$. While the physical world is describable in terms of $SU(1,1)$, we carry out the Iwasawa decomposition in the $Sp(2)$ regime.

Indeed, the conjugate transformation from Eq. (19) to Eq. (21) is from $SU(1,1)$ to $Sp(2)$, while the transition from Eq. (21) to Eq. (23) is within the $Sp(2)$ group. Thus, the transition from Eq. (19) to Eq. (23) is a conjugate transformation from the $SU(1,1)$ subgroup to the subgroup $Sp(2)$ of $SL(2,r)$.

Next, the mathematical instrument given in Sec. III is the decomposition of the $Sp(2)$ and $SU(1,1)$ matrices. Unlike the traditional approach to group theory which starts from the generators of the Lie algebra, we used in this paper an approach similar to what Goldstein did for the three-dimensional rotation group in terms of the Euler angles [13]. There are three generators for the rotation group, but Goldstein starts with rotations around the *z* and *x* directions. Rotations around the *y* axis and the most general form for the rotation matrix can be constructed from repeated applications of those two starting matrices. Let us call this type of approach the ''Euler construction.''

There are three basic advantages of this approach. First, the number of ''starter'' matrices is less than the number of generators. For example, we need only two starters for the three-parameter rotation group. In our case, we started with two matrices for the three-parameter group $Sp(2)$ and also for $SU(1,1)$. Second, each starter matrix takes a simple form and has its own physical interpretation.

The third advantage can be stated in the following way. Repeated applications of the starter matrices will lead to a very complicated expression. However, the complicated expression can be decomposed into the minimum number of starter matrices. For example, this number is three for the three-dimensional rotation group. This number is also three for $SU(2)$ and $Sp(2)$. We call this the Euler decomposition. The present paper is based on both the Euler construction and the Euler decomposition.

Among the several useful Euler decompositions, the Iwasawa decomposition plays the central role in this paper. In Sec. III we explained what the decomposition does to the two-by-two matrices of $Sp(2)$, but it has been an interesting subject since Iwasawa's first publication on this subject [8]. It is beyond the scope of this paper to present a historical review of the subject. However, we would like to point out that there are areas of physics where this important mathematical theorem was totally overlooked. For instance, in particle theory, Wigner's little groups dictate the internal space-time symmetries of massive and massless particles that are locally isomorphic to $O(3)$ and $E(2)$, respectively [14]. The little group is the maximal subgroup of the Lorentz group whose transformations do not change the four-momentum of a given particle [15]. The $E(2)$-like subgroup for massless particles is locally isomorphic to the subgroup of $SL(2,c)$, which can be started from one of the matrices in Eq. (34) and the diagonal matrix of Eq. (10). Thus there was an underlying Iwasawa decomposition while the the $E(2)$-like subgroup was decomposed into rotation and boost matrices [16], but the authors did not know this. One of those authors is one of the authors of the present paper.

In optics, there are many two-by-two matrices with one vanishing off-diagonal element. It was generally known that this has something to do with the Iwasawa effect, but Simon and Mukunda [9] and Han *et al.* [10] started treating the Iwasawa decomposition as the main issue in their papers on polarized light.

In para-axial lens optics, the matrices of the form given in Eq. (34) are the starters [17], and repeated applications of those two starters will lead to the most general form of $Sp(2)$ matrices. It had been a challenging problem since 1985 [17] to write the most general two-by-two matrix in lens optics in terms of the minimum number of those starter matrices. This problem has been solved recently [18], and the central issue in the problem was the Iwasawa decomposition.

In laser physics, there are many matrices of the form given in Eq. (34) with complex parameters [19,20]. Indeed, the Iwasawa decomposition appears to have a bright future in optics.

[1] D. Han, Y. S. Kim, and M. E. Noz, J. Opt. Soc. Am. A **14**, 2290 (1997); for earlier papers on this subject, see R. Barakat, J. Opt. Soc. Am. **53**, 317 (1963); C. S. Brown and A. E. Bak, Opt. Eng. **34**, 1625 (1995).

[2] D. Han, Y. S. Kim, and M. E. Noz, Phys. Rev. E **56**, 6065 (1997).

[3] R. A. M. Azzam and I. Bashara, *Ellipsometry and Polarized Light* (North-Holland, Amsterdam, 1977).

[4] J. J. Monzón and L. L. Sánchez-Soto, Am. J. Phys. **64**, 156 (1996).

[5] I. J. Lalov and E. M. Georgieva, J. Mod. Opt. **44**, 265 (1997).

[6] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).

[7] Y. S. Kim, J. Opt. Soc. Am. B **2**, R1 (2000); Y. S. Kim and M. E. Noz, *Phase Space Picture of Quantum Mechanics* (World Scientific, Singapore, 1991).

[8] K. Iwasawa, Ann. Math. **50**, 507 (1949); R. Hermann, *Lie Groups for Physicists* (Benjamin, New York, 1966).

[9] R. Simon and N. Mukunda, J. Opt. Soc. Am. **15**, 2146 (1998).

[10] D. Han, Y. S. Kim, and M. E. Noz, Phys. Rev. E **60**, 1036 (1999).

[11] J. J. Monzón and L. L. Sánchez-Soto, Phys. Lett. A **262**, 18 (1999).

[12] J. J. Monzón and L. L. Sánchez-Soto, J. Opt. Soc. Am. A **17**, 1475 (2000).

[13] H. Goldstein, *Classical Mechanics*, 2nd ed. (Addison-Wesley, Reading, MA, 1980).

[14] E. Wigner, Ann. Math. **40**, 149 (1939).

[15] For applications of the little groups in decoherence problems in optics, see D. Han, Y. S. Kim, and M. E. Noz, Phys. Rev. E **61**, 5907 (2000).

[16] D. Han, Y. S. Kim, and D. Son, J. Math. Phys. **28**, 2373 (1987); the title of this paper is *Decomposition of Lorentz Transformations*, but the authors did not know they were doing an Iwasawa decomposition.

[17] E. C. G. Sudarshan, N. Mukunda, and R. Simon, Opt. Acta **32**, 855 (1985).

[18] S. Baskal and Y. S. Kim, Phys. Rev. E **63**, 056606 (2001).

[19] H. Kogelnik and T. Li, Appl. Opt. **5**, 1550 (1966), and the references listed in this review paper.

[20] M. Nakazawa, J. H. Kubota, A. Sahara, and K. Tamura, IEEE J. Quantum Electron. **34**, 1075 (1998).